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PRACTICAL CALCULATIONS OF TRANSITIONAL BOUNDARY LAYERS

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Abstract—A general finite-difference procedure for computing the behavior of compressible two-dimensional boundary layers is presented together with a turbulence model which allows quantitative predictions of the location and extent of the transition region between laminar and turbulent flow as it is influenced by such disturbances as surface roughness and free-stream turbulence. Reverse transition, i.e. relaminarization, caused by large favorable streamwise accelerations, is also quantitatively predicted by this procedure. The solution procedure depends upon the calculation of the streamwise development of a turbulent mixing length whose magnitude is governed by the turbulence kinetic energy equation. A large number of comparisons between predictions and measurements have been made and in general very good agreement is obtained.

NOMENCLATURE

a_n, structural coefficients, equation (10);
C_p, specific heat at constant pressure;
C_f, skin friction coefficient $2\tau_w/\rho_e u_e^2$;
$\mathscr{D}$, sublayer damping factor;
E, turbulent kinetic energy equation source term, equation (9);
F, $\int_0^\eta (1. - \bar{\rho}\bar{u}/\bar{\rho}_e\bar{u}_e)\,d\eta$, stream function variable;
G, $\int_0^\eta (T_e^0 - T^0)/(T_e^0 - T_{ref})\,d\eta$, stagnation temperature defect integral;
H, shape parameter δ^*/θ;
$\bar{k}$, thermal conductivity;
k_T, turbulent effective thermal conductivity;
k^+, nondimensional roughness height $k_r\sqrt{(\tau/\bar{\rho})}/\nu$;
k_r, rms roughness height;
L, dissipation length, equation (10);
l, mixing length, equation (10);
P, pressure;
Pr, laminar Prandtl number;
Pr_T, turbulent effective Prandtl number;
Q, total effective heat flux, equation (6);
$\overline{q^2}$, total turbulent intensity, $\overline{u'^2} + \overline{v'^2} + \overline{w'^2}$;
R_θ, Reynolds number based on θ;
R_τ, turbulent Reynolds number, $\bar{\nu}_T/\bar{\nu}$;
T^0, stagnation temperature;
T_{ref}, arbitrary reference temperature;
T_u, free stream turbulence level $(\overline{q^2}/3)^{\frac{1}{2}}/u_e \times 100$;
u, local streamwise velocity;

$-\overline{u'v'}$,	Reynolds turbulent effective shear stress;
v,	velocity normal to the stream;
x,	streamwise length;
y,	height above the wall;
y^+,	nondimensional wall height, $y\sqrt{(\tau/\bar{\rho})}/\bar{v}$.

Greek symbols

β,	density ratio, ρ_e/ρ;
γ,	intermittency factor;
$\Delta\mathscr{D}_k$,	incremental sublayer damping factor due to roughness;
δ,	$u = 0{\cdot}99u_e$ boundary-layer thickness;
δ^+,	arbitrary thickness scale;
δ_s,	sublayer thickness;
δ_{th},	thermal boundary-layer thickness;
ϵ,	turbulent dissipation;
η,	nondimensional transverse distance, y/δ^+;
θ,	boundary-layer momentum thickness;
κ,	von Kármán constant;
μ,	viscosity coefficient;
ν,	kinematic viscosity, μ/ρ;
ν_T,	turbulent effective kinematic viscosity;
ρ,	density;
τ,	shear stress;
ϕ_1, ϕ_2, ϕ_3,	integral thicknesses, equations (12)–(14).

Subscripts

e,	free stream value;
w,	wall value.

Superscripts

$'$,	fluctuating quantity;
—,	time mean value.

INTRODUCTION

With the advent of large high speed digital computers it is now practical and, in view of the generality of the resulting solutions, desirable in certain instances to solve the boundary-layer partial differential equations of motion by a direct numerical approach. Such a direct approach is particularly attractive for predicting the behavior of transitional boundary layers since the boundary-layer partial differential equations can be similarly expressed for laminar, transitional, and fully-turbulent boundary layers. Consequently, one computer program can calculate the required solutions if the appropriate relationship between stress and rate of strain is inserted. From an engineering point of view, it seems that when the boundary-layer equations are valid it is possible to predict either the completely laminar or fully-turbulent boundary-layer development to a satisfactory degree of accuracy in a large number of flow situations of practical interest [1]. However, in many flows, such as the boundary-layer on a gas turbine airfoil of moderate Reynolds numbers [2], for at least some part of the development the boundary-layer is neither completely laminar nor fully turbulent.

To date, there has been very little progress in developing even simple semi-empirical theories which can predict the location and mean flow behaviour during transition (e.g. [3]). Shortly after starting this study the present authors became aware of a similar approach taken by Glushko [4] and later by Donaldson [5]. However, neither the work of Glushko nor that of Donaldson has been developed into a practical prediction system although Beckwith and Bushnell [6] have indicated that perhaps Glushko's approach could yield satisfactory engineering predictions after a short development period. More recently, Harris [7] has computed the transitional behavior of certain high Mach number boundary layers starting from a known transition location and using the suggestion of Dhawan and Narasimha [8] to determine the extent of the transition region. Harris' calculations showed a very encouraging degree of agreement with the measurements; however, as formulated, the approach has a very heavy reliance on empirical information and cannot *a priori* predict the effect of disturbances such as free-stream turbulence. At the time the present study was commenced, little beyond simple data correlations was available and the aim was, therefore, to provide an improved practical method of computing the development of the boundary-layer mean flow during transition which could take into account boundary conditions such as free-stream turbulence and wall roughness.

THEORY

The basic equations

Within the framework of the usual boundary-layer approximations, various authors, for example Schubauer and Tchen [9], have reduced the time-averaged Navier–Stokes equations to the compressible boundary-layer equations of motion. In the boundary-layer equations, it is convenient to represent the turbulent stress contribution to the total shear stress, τ, in terms of an effective turbulent viscosity, ν_T, and the turbulent temperature correlation contribution to the total heat flux, Q, in terms of an effective turbulent conductivity, k_T, where

$$\bar{\rho}\nu_T \partial\bar{u}/\partial y = -\bar{\rho}\overline{u'v'}; \; k_T\partial\bar{T}/\partial y = -\bar{\rho}C_p\overline{v'T'}. \quad (1)$$

The prime denotes a fluctuating quantity found in turbulent flow and the bar denotes a time mean-average. The effective turbulent conductivity is now related to the effective turbulent viscosity by the introduction of the turbulent Prandtl number defined by

$$Pr_T = C_p\bar{\rho}\nu_T/k_T. \quad (2)$$

When the turbulent and molecular Prandtl numbers are introduced and, in addition, the usual assumptions are made that the contributions from the longitudinal gradient of the Reynolds normal stress and normal pressure gradients are negligible, then for steady two-dimensional flow the boundary-layer equations, together with the continuity equation, may be written in the form

$$\bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial x} + \overline{\rho v}\frac{\partial\bar{u}}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial}{\partial y}\left\{(\bar{\mu} + \bar{\rho}\nu_T)\frac{\partial\bar{u}}{\partial y}\right\} \quad (3)$$

$$\bar{\rho}\bar{u}C_p\frac{\partial\bar{T}^0}{\partial x} + \overline{\rho v}C_p\frac{\partial\bar{T}^0}{\partial y} = \frac{\partial}{\partial y}\left\{\left(\frac{\bar{\mu}}{Pr} + \frac{\bar{\rho}\nu_T}{Pr_T}\right)C_p\frac{\partial\bar{T}^0}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{\left[\left(1 - \frac{1}{Pr}\right)\bar{\mu} + \left(1 - \frac{1}{Pr_T}\right)\bar{\rho}\nu_T\right]\bar{u}\frac{\partial\bar{u}}{\partial y}\right\} \quad (4)$$

$$\frac{\partial}{\partial x}(\bar{\rho}\bar{u}) + \frac{\partial}{\partial y}(\overline{\rho v}) = 0 \quad (5)$$

where the stagnation temperature $\bar{T}^0$, total apparent stress $\bar{\tau}$, and total effective heat flux Q, are defined as

$$\bar{T}^0 = \bar{T} + \bar{u}^2/2C_p, \; \bar{\tau} = \bar{\mu}\partial\bar{u}/\partial y - \bar{\rho}\overline{u'v'}, \quad Q = \bar{k}\partial\bar{T}/\partial y - \bar{\rho}C_p\overline{v'T'}. \quad (6)$$

These equations are, of course, also valid for laminar flow when all the turbulent correlations are zero.

The wall and free-stream boundary conditions employed in the solution are

$$\begin{aligned} &y = 0 \quad \overline{\rho v} = (\overline{\rho v})_w, \; \bar{T}^0 = T_w \text{ or } \partial\bar{T}/\partial y = 0 \\ &y \to \infty \quad \bar{\rho}\bar{u} = \bar{\rho}_e\bar{u}_e, \; \bar{T}^0 = T_e^0, \; \partial\bar{u}/\partial y = 0, \; \partial\bar{T}^0/\partial y = 0 \end{aligned} \quad (7)$$

where the subscripts w and e denote the wall and free-stream values, respectively.

In order to predict the development of the mean velocity and temperature field it remains to specify the effective turbulent viscosity and turbulent Prandtl number in terms of the mean flow variables and this is described in the following section. The term $\overline{\rho v}$ is eliminated from the momentum and energy equations by application of the continuity equation.

The turbulence model

Derivation of the integral turbulence kinetic energy equation. The basic turbulence model used in the present work is described by McDonald and Camarata [10] and in the present note the extension required to compute the development of two-dimensional compressible transitional boundary layers is given. The turbulence model assumes the existence of a well-behaved one parameter mixing length profile normal to the wall and computes the streamwise development of this mixing length profile using an integral form of the turbulence kinetic energy equation. In fully-turbulent flow, this procedure was found to result in quite accurate predictions of the boundary-layer development.

The turbulence kinetic energy equation is derived from the Navier–Stokes equations and in compressible flow, this derivation has been given by Favre [11]. After the introduction of certain boundary-layer approximations which restrict the use of this theory to nonhypersonic boundary layers (Bradshaw [12]), the turbulence kinetic energy equation can be integrated with respect to y to yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \overline{\rho u q^2}\,\mathrm{d}y = \int_0^\delta -\overline{\rho u'v'}\frac{\partial \bar{u}}{\partial y} - \int_0^\delta \bar{\rho}\epsilon\,\mathrm{d}y - \int_0^\delta \bar{\rho}(\overline{u'^2} - \overline{v'^2})\frac{\partial \bar{u}}{\partial x}\,\mathrm{d}y + E \quad (8)$$

where

$$\overline{q^2} = \overline{u'^2} + \overline{v'^2} + \overline{w'^2}$$

$$E = \left[\frac{1}{2}\overline{q^2}\left(\overline{\rho u}\frac{\mathrm{d}\delta}{\mathrm{d}x} - \overline{\rho v}\right) - \overline{p'v'} + \frac{1}{2}\overline{\rho q^2 v'} - \frac{1}{2}\overline{\rho' q^2 v'}\right]_e \quad (9)$$

and e represents the sum of the turbulent dissipation terms.

Following Townsend [13] and Bradshaw *et al.* [12], structural scales a_n and L are introduced, together with mixing length l, and these scales are defined as

$$-\overline{u'v'} = a_1\overline{q^2}, \overline{u'^2} = a_2\overline{q^2}, \overline{v'^2} = a_3\overline{q^2}$$
$$\epsilon = (-\overline{u'v'})^{\frac{3}{2}}/L, (-\overline{u'v'})^{\frac{1}{2}} = l\partial\bar{u}/\partial y. \quad (10)$$

The structural coefficients a_n are generally ascribed constant values independent of both x and y. This point will be discussed subsequently, but, if for the moment, it is accepted and integral parameters ϕ_1, ϕ_2, and ϕ_3 are introduced, equation (8) can be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi_1\bar{\rho}_e\bar{u}_e^3\delta^+}{2a_1}\right) = \bar{\rho}_e\bar{u}_e^3(\phi_2 - \phi_3 + E) \quad (11)$$

where

$$\phi_1 = \int_0^{\delta/\delta^+} \frac{\overline{\rho u}}{\rho_e u_e}\left(\frac{l}{\delta^+}\frac{\partial \bar{u}/\bar{u}_e}{\partial \eta}\right)^2 \mathrm{d}\eta \quad (12)$$

$$\phi_2 = \int_0^{\delta/\delta^+} \frac{\bar{\rho}}{\bar{\rho}_e}\left(\frac{l}{\delta^+}\right)^2\left(\frac{\partial \bar{u}/\bar{u}_e}{\partial \eta}\right)^3 (1 - l/L)\,\mathrm{d}\eta \quad (13)$$

$$\phi_3 = \int_0^{\delta/\delta^+} \frac{\bar{\rho}}{\bar{\rho}_e}\left(\frac{a_2 - a_3}{a_1}\right)\left(\frac{l}{\delta^+}\frac{\partial \bar{u}/\bar{u}_e}{\partial \eta}\right)^2 \frac{\delta}{u_e}\frac{\partial \bar{u}}{\partial x}\,\mathrm{d}\eta \quad (14)$$

where η is a nondimensional transverse distance y/δ^+ (δ^+ is arbitrary) and δ is the boundary-layer thickness. Equation (11) without the normal stress term ϕ_3 and E was used by McDonald and Camarata [10] to compute the downstream development of the mixing length l.

In equation (11), ϕ_1 is termed the convection integral, ϕ_2 the net production integral, and ϕ_3 the normal stress production integral. The definition of turbulence scales a_n, l, and L, only becomes questionable when restrictions are placed on the relationship of the scales to the other flow variables. The required, and it is thought plausible, relationships between the turbulence scales and the mean flow variables are now introduced. This process is greatly facilitated by introducing an hypothesis put forth by Morkovin [14], who suggested that, provided the fluctuation in Mach number remains small (which is true for free-stream Mach numbers below about 5 [12, 14]), the structure of turbulence remains essentially unaltered by the fluid compressibility. On the basis of this hypothesis, it is permissible to attribute incompressible values to the structural scales defined by equation (10). These structural scales are now considered in detail.

The dissipation length scale L. The definition of a turbulence length scale L in the manner given by equation (10) is based upon the results of isotropic turbulence. Past experience in shear flows leads one to expect that this scale would be reasonably constant in a boundary layer when expressed as a fraction of the overall boundary-layer thickness. An empirical fit to Bradshaw, *et al.*'s measurements [15] of L is given by McDonald and Camarata [10] as

$$L = 0{\cdot}1\,\delta \tanh\left[\kappa y/(\cdot 1\delta)\right] \qquad (15)$$

and κ is the von Kármán constant, taken as 0·43 throughout the present study. Within the viscous sublayer both the dissipation length L and the mixing length l are scaled by a damping factor $\mathscr{D}$ which will be discussed in detail subsequently. Outside the sublayer equation (15) has been used throughout, even for transitional boundary layers. The justification for taking an expression appropriate to a fully-developed turbulent boundary-layer to describe the dissipation length scale of transitional turbulence resides in the turbulence kinetic energy equation, equation (11), where it can be seen that, in the very early stages of transition when the Reynolds stress is still small, it follows that l, the mixing length, will also be small and, consequently, the net production integral ϕ_2 will be insensitive to the precise value of the dissipation length L, provided that L is not also small in this region. It is, of course, unlikely that transitional turbulence, being a weak low frequency disturbance, could have a small dissipation length scale, since a small dissipation length scale (i.e. large dissipation) is usually associated with a high frequency fluctuation.

The mixing length 1. Turning to the mixing length l defined by equation (10), McDonald and Camarata [10] pointed out that, on the basis of the experimental evidence, the mixing length distribution across turbulent boundary layers could be expressed as a one parameter family

$$l/\delta = l_\infty/\delta \tanh(\kappa y/l_\infty). \qquad (16)$$

For equilibrium boundary layers, l_∞/δ has a value near 0·09, but for nonequilibrium boundary layers l_∞/δ varies in the streamwise direction (Goldberg [16]). Insofar as transitional boundary layers are concerned there is little available in the way of measurements to provide guidance and, consequently, equation (16) was used even in the transitional region. As mentioned previously, within the viscous sublayer the mixing length is scaled by the damping factor $\mathscr{D}$ which will be described in detail subsequently.

When further information becomes available, more sophisticated n-parameter mixing length profiles can be introduced, requiring additional n-1 equations, which could certainly be obtained by taking integral moments of the turbulence kinetic energy equation. It is, however, a very convenient feature of equation (16) that, since the mixing length profile is specified in terms of a single additional parameter l_∞, only one additional equation, the integral turbulence kinetic energy equation, is required to describe the streamwise development of the turbulent shear stress.

The sublayer damping factor $\mathscr{D}$. It is well known that within the viscous sublayer (say, $y^+ < 50$, where $y^+ = y\sqrt{(\tau/\rho)}/\bar{\nu}$) the mixing length (and the dissipation length) go to zero as the wall is approached (see [17]). To model this effect the assumed mixing length profiles are scaled by a damping factor $\mathscr{D}(y^+)$. Various suggestions for this damping factor have been made (e.g. [17]) and the one used in the present note is based upon the notion that the flow in the viscous sublayer is only intermittently fully tubulent (with $l = \kappa y$ for instance, holding within the turbulent bursts). If it is then assumed that the mean flow gradient $\partial\bar{u}/\partial y$ is the same within and outside the turbulent patches, it can be argued that

$$-\overline{u'v'} = \gamma(-\overline{u'v'})_T = \gamma(l\partial\bar{u}/\partial y)_T^2 = \gamma l_T^2(\partial\bar{u}/\partial y)^2$$
$$= (l_T\partial\bar{u}/\partial y\mathscr{D})^2 \qquad (17)$$

where γ is the intermittency factor and the subscript 'T' refers to the fully-turbulent patches.

It follows from equation (17) that the conventionally defined damping factor $\mathscr{D}$, is simply the square root of the intermittency factor γ. In the present note, it is assumed that the damping is distributed normally around some mean height $\overline{y^+}$ (=23.) with a standard deviation $\sigma(=8.)$, resulting in a smooth wall damping factor given by

$$\mathscr{D}_s = P^{\frac{1}{2}}\{(y^+ - \overline{y^+})/\sigma\} \quad (18)$$

where P is the normal probability function. Excellent agreement with measured smooth wall sublayer mean velocity profiles is obtained using equation (18). In the present analysis y^+ is based upon a local stress; however, the damping was not allowed to decrease once a maximum value was achieved.

The effect of wall roughness on the turbulent boundary-layer has long been recognized as resulting mainly in a decrease in the additive constant of the law of the wall, leaving the fully-turbulent part of the motion largely unaltered. Van Driest [17] suggested that this effect could be modelled by progressively eliminating the sublayer damping as the roughness height is increased. However, a direct application of Van Driest's suggestion does not permit the additive constant to decrease further once the sublayer has been eradicated by a sufficiently large roughness height ($k^+ > 60$, where $k^+ = k_r\sqrt{(\tau/\bar{\rho})}/\bar{\nu}$ and k_r is the rms roughness height) in conflict with the experimental evidence. This deficiency is readily removed by allowing damping factors in excess of unity and it was found that an incremental damping factor due to roughness, $\Delta\mathscr{D}_K$ where $\Delta\mathscr{D}_K$ is given by

$$\Delta\mathscr{D}_K = (1. + k^+/30.y^+)\cdot\exp(-2{\cdot}3\, y^+/k^+) \quad (19)$$

and $\mathscr{D} = \mathscr{D}_s + \Delta\mathscr{D}_K$ resulted in very good agreement between predicted and measured additive constants up to roughness heights of order $k^+ = 10^4$. Equation (19) was the sole means of introducing the effect of roughness into the prediction procedure.

The structural coefficients a_n. The calculations presented in the present note are not at all sensitive to the assumed values of a_2 and a_3 (see equation (10)). The fully turbulent flow values for these coefficients of 0·5 and 0·2 (Bradshaw [15]; McDonald and Camarata [10]) were, consequently, used throughout. It would be expected that the coefficient a_1 (in the subsequent discussion the subscript '1' is dropped) would vary both with the applied strain $\partial\bar{u}/\partial y$ and, based on the experimental evidence of the viscous sublayer, with the relative stress level. It is, of course, to be expected that a would vary with the applied strain since in isotropic turbulence $\partial\bar{u}/\partial y$ is zero and so is the Reynolds shear stress. The departure of the Reynolds shear stress from zero and the growth of a is associated with alignment of the structure under the action of the applied strain. On the basis of the measured response of isotropic turbulence to applied strain it is thought that, in the absence of any viscous effects, the adjustments of a to its fully-strained value would occur much faster than transition to turbulent flow. Initial calculations, therefore, used a fully-strained value of $a = 0{\cdot}15$, however, it soon became apparent that plausible transition regions required a much lower value for a. The departure of a from the fully-turbulent value was ascribed solely to the action of viscosity and to quantify this relationship it is necessary to introduce an appropriate Reynolds number, R_τ. If, in the definition of this Reynolds number, R_τ, the velocity scale is taken as the square root of the Reynolds stress, $-\overline{u'v'}$, and the length scale as the conventional mixing length, then the turbulence Reynolds number, R_τ, is simply the ratio of turbulent contribution to the apparent viscosity, ν_T, to the actual viscosity of the fluid, $\bar{\nu}$, i.e.

$$a = f(R_\tau) = f(\bar{\nu}_T/\bar{\nu}). \quad (20)$$

To simplify the solution of the turbulence kinetic energy equation, equation (11), at a given streamwise location a layer-averaged value of R_τ $\bar{R}_\tau$ is used, and defined in straightforward manner as

$$\overline{R_\tau} = \frac{1}{\delta}\int_0^\delta \bar{\nu}_T\,dy \bigg/ \frac{1}{\delta_s}\int_0^{\delta_s} \bar{\nu}\,dy \quad (21)$$

where δ_s, the sublayer thickness, is defined as the location at which the laminar stress has fallen to 4 per cent of the total stress. In this way the viscosity in the definition of the turbulence Reynolds number is taken to be the average viscosity in the region where viscosity could influence the mean flow development. Several alternate definitions of the turbulence Reynolds number were examined and the predictions were not found to be critically dependent on the choice of definition. Assuming the turbulence Reynolds number $\bar{R}_\tau$ is the sole variable influencing the structural coefficient a, then it is only necessary to derive the relationship between the two parameters for one set of flow conditions to have it hold for all flow conditions. Based upon observation of the incompressible constant pressure flat plate equilibrium turbulent boundary-layer, it is readily ascertained that in the turbulence kinetic energy equation ϕ_3, E, and the streamwise derivative of ϕ_1 are negligible for this flow, resulting in a reduced turbulence kinetic energy equation which can be written as a Bernoulli equation

$$\mathrm{d}\,(\ln a)/\mathrm{d}R_\theta = f(R_\theta) + g(R_\theta)\,.\,a \qquad (22)$$

with $f = \mathrm{d}\ln\delta/\mathrm{d}R_\theta$, $g = -4\phi_2\theta/R_\theta C_f\phi_1\delta$

where C_f is the skin friction coefficient and for convenience the independent variable has been changed from the streamwise distance x to the Reynolds number based on momentum thickness using the momentum integral equation. The arbitrary scale length δ^+ has been taken as the boundary-layer thickness δ. Equation (22) has the solution

$$a = [\exp\textstyle\int f\,\mathrm{d}R_\theta]/[C - \int g \exp\int f\,\mathrm{d}R_\theta \mathrm{d}R_\theta] \qquad (23)$$

which, upon specification of the relationship between $\bar{R}_\tau$, f, g, and R_θ for the equilibrium flat plate boundary-layer, provides the required general relationship between a and $\bar{R}_\tau$. Taking $f(R_\theta)$ first, it is noted that the reasonable assumption that θ/δ does not vary greatly with momentum thickness Reynolds number gives f as $1/R_\theta$. Furthermore, the group of terms $4\phi_2\theta/(C_f\phi_1\delta)$ was evaluated by numerically integrating the profiles of Maise and McDonald [19] and for a wide range of Reynolds numbers (these results also verified the neglect of the streamwise derivative of ϕ_1 in deriving equation (23)). It was found that the grouping $4\phi_2\theta/(C_f\phi_1\delta)$ was sensibly constant with a mean value of 6·66, giving $g = -6{\cdot}66/R_\theta$. With these relationships for f and g, equation (23) reduces to

$$a = a_0(R_\theta/R_{\theta_0})/[1\cdot + 6{\cdot}666a_0(R_\theta/R_{\theta_0} - 1)] \quad (24)$$

where a_0 is the value of a at a specified Reynolds number R_{θ_0}. It is observed for large R_θ/R_{θ_0} that equation (24) asymptotes out to give $a = 0{\cdot}15$, the expected value. To change independent variables from R_θ to $\bar{R}_\tau$, once again recourse is made to the profiles of Maise and McDonald [18] which integrate to yield for the higher Reynolds number range

$$R_\theta = 68{\cdot}1\,\bar{R}_\tau + 614{\cdot}3 \qquad \bar{R}_\tau > 40. \qquad (25)$$

Little information is available on the $R_\theta - \bar{R}_\tau$ relationship at low R_θ. On the basis of trial and error it was found that

$$R_\theta = 100\,.\,\bar{R}_\tau^{0{\cdot}22} \qquad \bar{R}_\tau \leqslant 1 \qquad (26)$$

gave excellent results, and, in the intermediate zone between the two regimes, a simple cubic was constructed to match value and slope at the join points. Finally, on the basis of a comparison between theory and experiment, the constant of integration in equation (24) was selected and resulted in a value for a_0 of 0·012 at a $\bar{R}_\tau$ of unity. For all of the calculations presented in the present note, the foregoing relationships were fixed and not allowed to change from flow to flow.

Method of solution

The first step in transforming the partial differential equations to a form more convenient for computer solution was to introduce the following new variables

$$\eta = y/\delta^+, F'(\eta) = 1{\cdot}0 - \overline{\rho u}/\bar{\rho}_e\bar{u}_e,$$
$$G'(\eta) = (\bar{T}_e^0 - \bar{T}^0)/(\bar{T}_e^0 - T_{\text{ref}}), \beta = \bar{\rho}_e/\bar{\rho} \qquad (27)$$

where the prime denotes differentiation with respect to η. The resulting momentum and energy equations are third order in $F(\eta)$ and $G(\eta)$, respectively. The scaling of y by the thickness δ^+ allows the use of a fixed η grid normal to the wall. The chosen definition of $F'(\eta)$ in the form of a stream function allows easy introduction of the continuity equation. The equations were then reduced to ordinary differential equations normal to the wall by use of the Hartree–Womersley (see [19]) approach of replacing initially the x derivatives by finite differences. The resulting third order ordinary differential equations for $F(\eta)$ and $G(\eta)$ are linearized and each solved by implicit Gaussian elimination at the desired streamwise location, using the boundary conditions specified by equation (7). The nonlinear terms in the equations are then iteratively updated and the momentum and energy equations resolved until the wall stress and wall heat flux (or wall temperature if an adiabatic wall condition is specified) do not change within some specified tolerance. At this point the solution procedure is repeated at next streamwise location.

The role of free-stream turbulence and related topics

Free-stream turbulence is seen to provide a small source term E in the turbulence kinetic energy equation which starts the transition process. Items such as wall roughness, transpiration, etc., by changing the turbulence production mechanisms within the boundary-layer, alter the magnitude of the Reynolds stress at a given streamwise location, and hence move the transition region. One aspect of the effect of roughness missing from the present analysis is its effect on the laminar boundary-layer. However, experimentally, little effect of roughness on the laminar boundary-layer is observed providing the roughness height is less than about half the boundary-layer displacement thickness.

In the source term E, equation (9), usually only the $\overline{q^2}$ term has any real contribution. However, since it is well-known that energy can usually only be exchanged between waves of similar frequencies, it is to be expected that the frequency as well as the intensity of the free-stream turbulence would have an influence on the transition region. Consequently, in applying equation (9) the turbulence intensity should be interpreted as the intensity of "active" free-stream turbulence; that is, the free-stream turbulence made up largely of frequencies similar to those expected in the boundary-layer turbulent shear stress spectrum. In the calculations performed to date, the "active" free-stream turbulence has been taken as the nominal value quoted by the experimentor with one exception which is pointed out.

In cases where the local free stream velocity is changing and the free stream turbulence is quoted at only one streamwise location, the additional assumption of frozen turbulence is made to enable the development of the source term E to be computed. Simply stated frozen turbulence assumes that the absolute fluctuation level remains unaltered by a change in the free stream velocity. This assumption should be reasonable for flows where the free stream changes occur rapidly and the mean flow transverse gradients are small.

COMPARISON WITH EXPERIMENTS

Since the turbulence model used herein reduces to the method of Spalding and Patankar [20], including the value of empirical constants, when the upstream history effect is negligible, the results from the present method are at least as accurate as those computed by Spalding and Patankar for fully-developed near equilibrium boundary layers. The comparisons presented herein demonstrate the additional capability of present procedure to predict the location, extent, and behavior of a variety of transitional flows. In related studies, Shamroth and McDonald [21] compare the transitional predictions from the present analysis with measurements at low hypersonic speeds, while Briley and McDonald [22] have applied the present analysis to transitional separation bubbles.

Effect of free-stream turbulence

In Fig. 1 the measured displacement thickness Reynolds numbers at transition with varying free-stream turbulence levels from a wide range

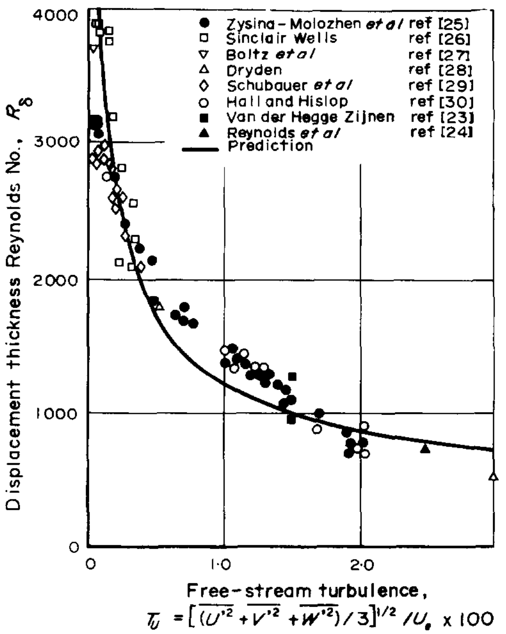

FIG. 1. Effect of free-stream turbulence upon the transition Reynolds number.

of experiments are shown compared with the predictions of the present analysis. Insofar as the analysis is concerned, the transition point is defined as the minimum skin friction location and all the calculations were initiated from a front stagnation point. Once the immediate stagnation region was negotiated, the turbulence kinetic energy equation was integrated in collaboration with the streamwise momentum equation. At the first streamwise station a mixing length of $0{\cdot}0001\delta$ was assumed to initiate the calculation. The transition location is quite insensitive to this initial value provided it is small and nonzero. In Fig. 2 the measured incremental Reynolds numbers from the beginning to the end of transition are shown plotted against the transition Reynolds number, the

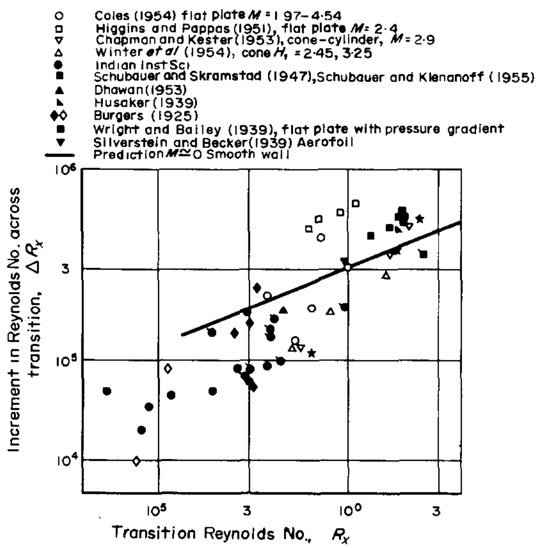

FIG. 2. Variation of transition length with transition Reynolds number.

data being reproduced from Dhawan and Narasimha [8]. The experimental definition of the beginning and end of transition used by Dhawan and Narasimha is not clear. Also shown in Fig. 2 are the incremental Reynolds numbers obtained from the predictions of the effect of free-stream turbulence used to construct Fig. 1. To obtain the predicted increments the skin friction was plotted against Reynolds number and in the transition zone a straight line approximation made to the skin friction curve. Intersection of this straight line with the laminar and turbulent lines defined the beginning and end of transition. Although the scatter in the data is considerable, the predicted trend and level obviously quite reasonable. In Fig. 3 a comparison with the measured Stanton numbers of Reynolds *et al.* [24], is shown and excellent agreement demonstrated. In Fig. 4 a detailed comparison with the measured transitional mean velocity profiles of Schubauer and

Klebanoff [31] is shown. For the purposes of this comparison the analysis was carried out from a front stagnation point using a higher value of the free-stream turbulence level than quoted in the experiment in order to initiate transition at the experimentally observed point,

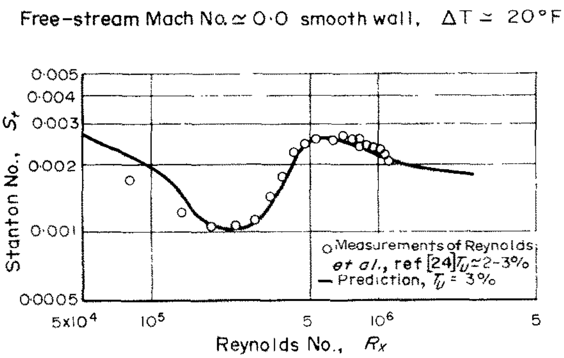

FIG. 3. A comparison between measured and predicted Stanton number through transition.

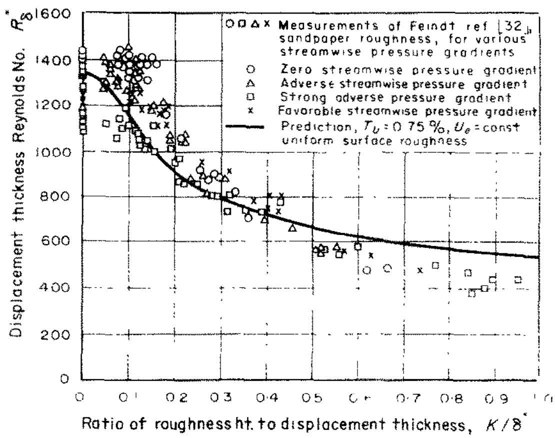

FIG. 5. Effect of distributed surface roughness upon transition Reynolds number.

of the present analysis for the displacement thickness Reynolds number at transition as a function of the roughness height normalized by

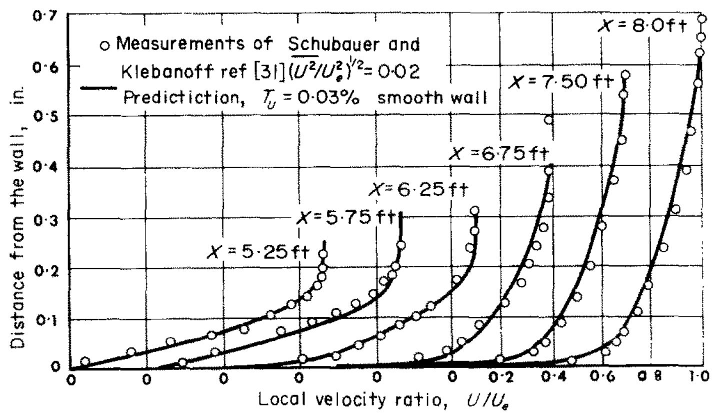

FIG. 4. Comparison between measured and predicted mean velocity profiles during transition.

but this should not significantly affect the validity of the comparison with the subsequently developed mean velocity profiles during transition.

Effect of wall roughness

In Fig. 5 a comparison is presented between Feindt's [32] measurements and the predictions the displacement thickness at transition. In Fig. 6 a detailed comparison between the measured and predicted mean velocity profiles for a particular transitional boundary layer measured by Feindt is presented. All the roughness calculations were made using a value of 0·75 per cent for the free-stream turbulence and all were started from a front stagnation point.

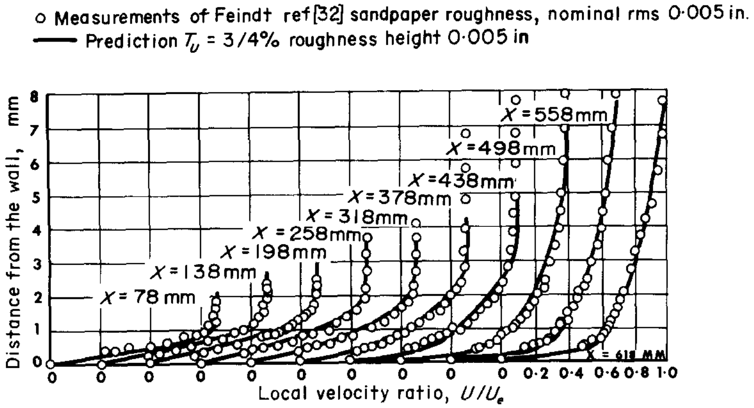

FIG. 6. Comparison between measured and predicted mean velocity profiles during transition.

The roughness was introduced once the immediate stagnation region was negotiated.

Effect of large streamwise acceleration

The effect of a large streamwise acceleration upon a turbulent boundary-layer has been studied both experimentally and theoretically by a number of investigators, for example, Launder and Jones [33]. There can now be little doubt that a sufficiently large favorable streamwise velocity gradient (say, $K \sim 10^{-6}$, where $K = v_w/u_e^2 \, \mathrm{d}U_e/\mathrm{d}x$), imposed for a sufficient period of the flow development, will cause a turbulent boundary-layer to become very laminar-like in appearance. Launder and Jones [33] point out that the self-preserving so-called "sink flow" boundary-layers obtained by imposing a constant value of K on the flow form a very convenient family of flows with which to investigate the relaminarization phenomenon. In both laminar and turbulent flow the "sink flow" boundary layers are characterized by a self-preserving mean velocity profile shape and a constant value of the momentum thickness Reynolds number. In Fig. 7 the completely laminar and fully-turbulent "sink flow" boundary-layer momentum thickness Reynolds number are presented as a function of the acceleration parameter K. The fully-turbulent line was computed using a constant outer layer

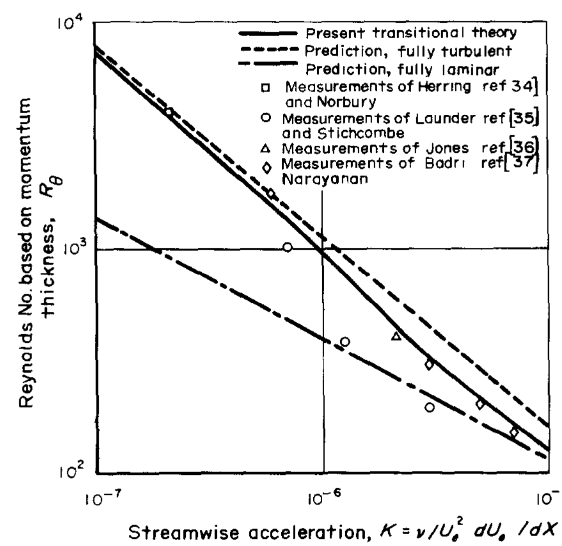

FIG. 7. Comparison between predictions and measurements for the "sink flow" boundary-layer Reynolds number.

mixing length of 0·09δ. Also shown in Fig. 7 is the computed variation according to the present study and a gradual departure from the turbulent line is observed when K is greater than about 10^{-6}. The experimental data collected by Launder and Jones [33] are also shown in Fig. 7. In Fig. 8 a comparison is presented between a measured boundary-layer growing on a nozzle wall (Nash-Weber [38]) and the predictions of the present analysis. Once again the improvement of the present prediction over a fully-turbulent constant mixing layer calculation is demonstrated.

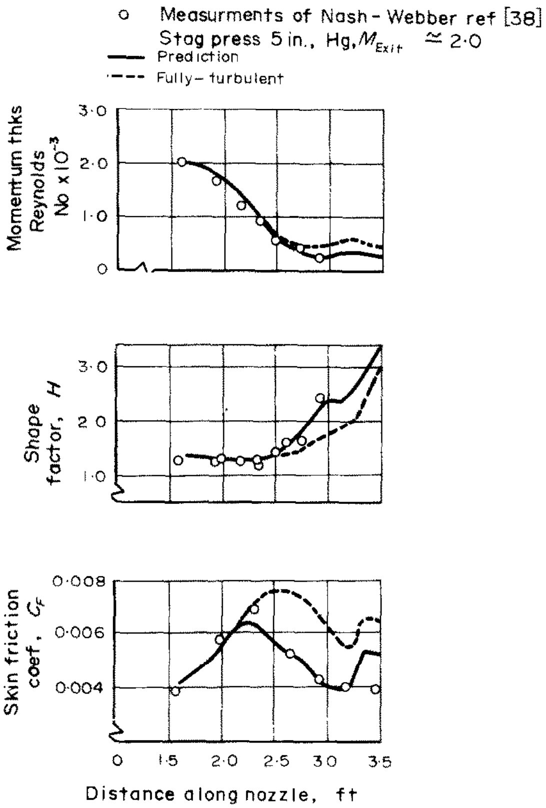

FIG. 8. A comparison between predictions and measurements for a superonic nozzle.

Transition on a turbine airfoil

Turner [2] has made measurements of the heat transfer distribution on a typical turbine airfoil with three free-stream turbulence levels. and noted some very substantial effects. To compare with Turner's data in Figs. 9 and 10 it was necesrary to estimate the stream velocity upon which the quoted free-stream turbulence levels were based. This entailed running a potential flow calculation for Turner's cascade to compute the entrance plane mid-channel velocity. Considering the uncertainties in determining the reference velocity in this manner and the inherent errors in the measurement of free-stream turbulence, a difference of ± 30 per cent between the free-stream turbulence level used in the present predictions and the actual value for the experiments is not unreasonable. Furthermore, in view of the fine balance between

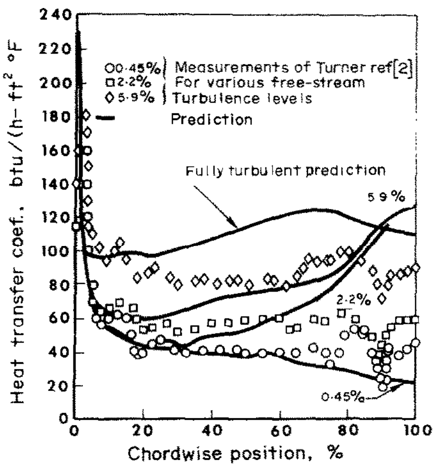

FIG. 9. Heat-transfer distribution on the pressure side of a turbine airfoil.

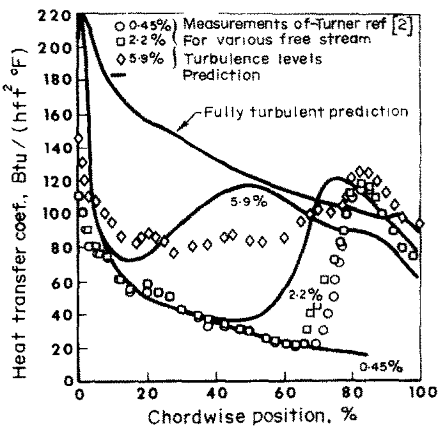

FIG. 10. Heat-transfer distribution on the suction side of a turbine airfoil.

streamwise acceleration and transition, the calculations are quite sensitive to the imposed pressure distribution. Consequently. any quantitative evaluation of the comparisons between the present predictions and experiment must be viewed with caution. Lastly, it is worth remembering that unless some special *ad hoc* procedure were constructed, conventional prediction methods would give either the completely laminar or fully-turbulent heat transfer distribution and to bridge the gap a great deal of intuition and previous experience would be required.

DISCUSSION

It was previously argued that in the very initial stages of transition the integral turbulence kinetic energy equation reduced to turbulence advection being approximately equal to turbulence production. To provide some further insight into the transition process, the crude assumption can be made that the local Reynolds stress terms in such a reduced energy equation may be replaced by y-averaged values. Thus, with the assumption of incompressible flow with a constant free-stream velocity, the turbulence kinetic energy equation can be written

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\overline{q^2_{\text{ave}}}(\delta - \delta^*)\right] = (-\overline{u'v'})_{\text{ave}}. \qquad (28)$$

The implication of equation (28) is that, given a disturbance field, such as free-stream turbulence, which is capable of producing some turbulent shear stress within the boundary-layer, then the turbulence intensity will not increase in the streamwise direction so long as the boundary-layer grows sufficiently rapidly to absorb the newly produced turbulence. However, it is well-known from momentum considerations that, in the absence of a streamwise pressure gradient, the rate of boundary-layer growth decreases with increasing Reynolds number. Consequently, at some Reynolds number the boundary-layer growth cannot absorb the newly produced turbulence, and so in order to maintain the integral energy balance the turbulence intensity increases in the streamwise direction and the transition process begins.

Thus far, no mention has been made of the relationship of the present work to stability theory. A formal relationship must, of course, exist, since the complete turbulence kinetic energy equation governs both the wave motion of stability theory and the developing turbulence. Using stability theory to predict transition is an effort to extrapolate forward from laminar flow into turbulent flow. The present approach is to predict transition by extrapolating backwards from turbulent flow. It is hardly surprising that the present state of the art is such that neither approach can properly be made to overlap. Neither approach has sufficient detail of the flow structure to follow the development of a simple wave into turbulence. The philosophy of the present approach is based upon the belief that in a large number of practical situations the development of simple unstable waves into turbulence occurs rapidly and the real question of transition concerns the advection–production balance of the turbulence. There seems little doubt that given the terms of reference of the present authors, which was to develop a practical method of calculating the position and mean flow development during transition, the approach adopted in the present note has proven much more rewarding than could have been optimistically expected from stability theory.

As an additional point, it should be recalled that, in certain carefully controlled laboratory experiments, it has been observed that, in the very initial stages, transition is a very well-organized three-dimensional phenomenon. Obviously, the use of a two-dimensional approach like the present method (albeit assuming a three-dimensional turbulence structure) must be questioned when regular spanwise variations are observed in the flow. However, it is noted that first that the mean flow usually is unaffected by the transition process at this early stage, and second the spanwise variations are usually only observed when the disturbance field is very weak. Generally speaking, in most practical environments the spanwise variations are either not observed, or occur over such a short streamwise extent as to be negligible. In any event, the present two-dimensional approach can be regarded as providing a spanwise averaged description of the transition process.

As a final remark, it is observed that the present approach places little emphasis on the frequency of the disturbance and only the mean disturbance energy is considered important. This is obviously an oversimplification in certain instances. However, it is noted experimentally that, for instance, when the free-stream turbulence intensities are greater than about 0·25 per cent, the resulting transition locations from a wide range of tunnels, measured by various experimentors over many years, depends solely on the free-stream turbulence energy level and apparently not on its frequency content. At turbulence levels of less than 0·25 per cent, it does appear as if a maximum transition Reynolds number for a given tunnel can be achieved (an acoustic phenomenon?) and further reduction in the free-stream turbulence level would be ineffective. The present analysis does not reflect this cutoff phenomenon and the predictions at these low turbulence levels must be regarded as upper limits of the transition Reynolds number.

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CALCULS PRATIQUES DE COUCHES LIMITES DE TRANSITION

Résumé—Une méthode générale aux différences finies pour le calcul du comportement de couches limites compressibles bidimensionnelles est présentée avec un modèle de turbulence qui permet des estimations quantitatives de la localisation et de l'étendue de la région de transition entre l'écoulement laminaire et turbulent quand elle est influencée par des perturbations telles que la rugosité superficielle et la turbulence de l'écoulement libre. Une transition inverse, c'est-à-dire la relaminarisation causée par de grandes accélérations favorables dans le sens de l'écoulement est aussi estimée quantitativement par cette méthode. La résolution dépend du calcul du développement dans le sens de l'écoulement d'une longueur de mélange par turbulence dont la grandeur est gouvernée par l'équation d'énergie cinétique de turbulence. Un grand nombre de comparaisons entre les estimations et mesures a été fait et de façon générale elles sont en bon accord.

PRAKTISCHE BERECHNUNG VON ÜBERGANGSGRENZSCHICHTEN

Zusammenfassung—Ein allgemeines Verfahren mit finiten Differenzen zur Berechnung des Verhaltens von kompressiblen zweidimensionalen Grenzschichten wird zusammen mit einem Turbulenz-Modell vorgestellt. Es gestattet quantitative Voraussagen über den Ort und das Ausmaß des Übergangsbereiches zwischen laminarer und turbulenter Strömung zu machen und wie diese von Störungen wie Oberflächenrauhigkeit und Freistromturbulenz beeinflusst werden. Der umgekehrte Übergang, d.h. die Rücklaminarisierung, hervorgerufen durch grosse günstige Beschleunigungen der Strömung, kann mittels dieses Verfahrens auch quantitativ vorausgesagt werden. Das Lösungsverfahren hängt ab von der Berechnung der Strömungsentwicklung und von einer turbulenten Mischungslänge, deren Größe durch die Gleichung für die kinetische Energie der Turbulenz bestimmt ist. Eine grosse Zahl von Voraussagen und Messungen wurde verglichen und im allgemeinen eine sehr gute Übereinstimmung festgestellt.

ПРАКТИЧЕСКИЕ РАСЧЕТЫ ПЕРЕХОДНЫХ ПОГРАНИЧНЫХ СЛОЕВ

Аннотация—Предложена математическая модель турбулентных течений и общий конечно-разностный метод расчета характеристик сжимаемых двумерных пограничных слоев, которые позволяют количественно рассчитать положение и размеры переходной области между ламинарным и турбулентным потоками при наличии воздействия шероховатости поверхности и турбулентности основного потока. С помощью этого метода можно также количественно рассчитать обратный переход, т.е. реламинаризацию потока, вызываемую большим сопутствующим ускорением в направлении движкния. Численная процедура зависит от расчета распределения в направлении движения длины пути смешения, величина которого описывается уравнением турбулентной кинетической энергии. Неоднократное сравнение результатов расчета с данными измерений показало очень хорошее соответствие сопоставленных данных.